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# THE FRACTIONAL CHROMATIC NUMBER OF THE CATEGORICAL PRODUCT OF GRAPHS

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We prove that the identity

$$\chi_f(G \times H) \geq \tfrac{1}{4} \cdot \min\{\chi_f(G), \chi_f(H)\}$$

holds for all directed graphs $G$ and $H$. Similar bounds for the usual chromatic number seem to be much harder to obtain: It is still not known whether there exists a number $n$ such that $\chi(G\times H)\geq 4$ for all directed graphs $G$, $H$ with $\chi(G)\geq\chi(H)\geq n$. In fact, we prove that for every integer $n\geq 4$, there exist directed graphs $G_n$, $H_n$ such that $\chi(G_n)=n$, $\chi(H_n)=4$ and $\chi(G_n\times H_n)=3$.

## 1. Introduction

The *categorical product* $G\times H$ of two directed graphs $G$ and $H$ is the directed graph with vertex-set $V(G)\times V(H)$ and arcs $(u_1,u_2)\to(v_1,v_2)$ for every $u_1\to v_1$ in $G$ and $u_2\to v_2$ in $H$. (Here, we write $u\to v$ to indicate the presence of an arc from $u$ to $v$.) The product $G\times H$ admits homomorphisms (that is, arc-preserving maps) to both $G$ and $H$. This allows one to give an upper bound for the fractional chromatic number of $G\times H$ in terms of its factors:

$$\chi_f(G \times H) \leq \min\{\chi_f(G), \chi_f(H)\}. \quad (1)$$

In this paper, we provide the following lower bound:

**Theorem 1.** *For any two directed graphs $G$ and $H$,*

$$\chi_f(G \times H) \geq \tfrac{1}{4} \cdot \min\{\chi_f(G), \chi_f(H)\}. \quad (2)$$

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In [9], we gave examples of $n$-tournaments $S_n$, $T_n$ such that $\chi(S_n \times T_n) \simeq \frac{2}{3} \cdot n$. The best possible lower bound for the fractional chromatic number of a product of directed graphs in terms of the the fractional chromatic numbers of the factors will be of the type $\chi_f(G \times H) \geq c \cdot \min\{\chi_f(G), \chi_f(H)\}$, for some $c$ between $\frac{1}{4}$ and $\frac{2}{3}$.

In contrast, it seems very difficult to find meaningful lower bounds for the chromatic number of a product of directed graphs in terms of the chromatic numbers of the factors. Poljak and Rödl [5] introduced the function

$$\phi(n) = \min\{\chi(G \times H) : G,\ H \text{ are directed graphs and } \chi(G) \geq \chi(H) \geq n\}.$$

It is not even known whether $\phi$ goes to infinity with $n$. In [4,11], it is shown that either $\phi(n) = \min\{3, n\}$ for all $n$, or $\lim_{n\to\infty} \phi(n) = \infty$. In fact the hypothesis that $\phi$ is bounded by 3 seems plausible in view of our next result:

**Theorem 2.** *Let $T_n$ denote the transitive tournament on $n$ vertices. Then there exists a 4-chromatic directed graph $H_n$ such that $\chi(T_n \times H_n) \leq 3$.*

If one used arbitrary graphs for both factors of the categorical product instead of restricting one factor to the relatively small class of transitive tournaments it seems plausible that Theorem 2 could be improved to families of directed graphs $\{G'_n\}_{n\in\mathbb{N}}$, $\{H'_n\}_{n\in\mathbb{N}}$ such that $\chi(G'_n) = n$, $\chi(H'_n) = k$ and $\chi(G'_n \times H'_n) \leq 3$. This holds for *all* values $k$ if and only if the function $\phi$ is bounded by 3.

Now undirected graphs can be viewed as symmetric directed graphs (where each edge corresponds to two opposite arcs). Hedetniemi's conjecture states that the bound

$$\chi(G \times H) \leq \min\{\chi(G), \chi(H)\} \tag{3}$$

is tight for all undirected graphs $G$, $H$. This conjecture has attracted a lot of attention; see the recent surveys [6,11]. In [12], the question as to whether the bound (1) is always tight for undirected graphs is discussed. If either of (1), (3) is tight for undirected graphs, then the bound

$$\chi(G \times H) \geq \min\{\chi_f(G), \chi_f(H)\} \tag{4}$$

holds for all undirected graphs $G$, $H$. In [8] the bound

$$\chi(G \times H) \geq \tfrac{1}{2} \min\{\chi_f(G), \chi_f(H)\} \tag{5}$$

is proved for directed graphs. We note that no improvements on the bounds (2), (5) are known for undirected graphs.

## 2. Proof of Theorem 2

Let $T_n$ be the transitive tournament with vertex-set $1,\dots,n$ and arcs $x\to y$ such that $x<y$. Let $H_n$ be the graph with vertex set

$$V(H_n) = \{f_{i,j} : i \in \{0,1,2\}, j \in \{1,2,\dots,n-1\}\},$$

where $f_{i,j}$ is the function from $V(T_n)$ to $V(K_3)=\{0,1,2\}$ defined by

$$f_{i,j}(x) = \begin{cases} i \text{ if } x \le j \\ i \oplus 1 \text{ if } x > j \end{cases}$$

(We denote by $\oplus$ the addition in $\mathbb{Z}_3$ and by $+$ the addition in $\mathbb{N}$). The arcs of $H_n$ are the couples $f_{i,j}\to f_{i',j'}$ such that $x\to y$ in $T_n$ implies $f_{i,j}(x)\neq f_{i',j'}(y)$. In other words, $f_{i,j}\to f_{i',j'}$ is an arc of $H_n$ if and only if $i'=i\oplus 1$ and $j'\le j+1$, or $i'=i$, $j=n-1$ and $j'=1$.

The first description of the arcs of $H_n$ allows one to exhibit a natural 3-colouring of $T_n\times H_n$: let $c\colon T_n\times H_n\mapsto\{0,1,2\}$ be defined by $c(x,f_{i,j})=f_{i,j}(x)$. If $(x,f_{i,j})\to(y,f_{i',j'})$ in $T_n\times H_n$, then $x\to y$ in $T_n$ and $f_{i,j}\to f_{i',j'}$ in $H_n$, whence $f_{i,j}(x)\neq f_{i',j'}(y)$; this shows that $c$ is a proper 3-colouring of $T_n\times H_n$.

We use the second description of the arcs of $H_n$ to show that $\chi(H_n)\ge 4$. Suppose that $H_n$ admits a proper 3-colouring $c\colon H_n\mapsto\{0,1,2\}$. Without loss of generality, we can suppose that $c(f_{i,1})=i$, $i=0,1,2$. Since $f_{i,j}\to f_{i\oplus 1,1}$, we then have $c(f_{i,j})\in\{i,i\oplus 2\}$, $j=1,\dots,n-1$, and in particular $c(f_{i,n-1})=i\oplus 2$. For $i=0,1,2$, put $j_i=\max\{j : c(f_{i,j})=i\}$; the structure of $H_n$ then implies $j_0<j_1<j_2<j_0$, which is impossible. Therefore, $\chi(H_n)\ge 4$. ∎

## 3. Proof of Theorem 1

The graph $H_n$ of the previous section is actually a subgraph of the exponential graph $K_3^{T_n}$, whose vertices are *all* the functions from $V(T_n)$ to $V(K_3)$. In general, for two graphs $G$ and $K$ (directed or undirected), the *exponential graph* $K^G$ is the graph whose vertices are all the functions from $V(G)$ to $V(K)$, where $f\to g$ if and only if for all $x\to y$ in $G$, we have $f(x)\to g(y)$ in $K$.

There is a natural correspondence between the homomorphisms from $G\times H$ to $K$ and the homomorphisms from $H$ to $K^G$: If $\psi : G\times H\mapsto K$ is a homomorphism, then the map $\hat{\psi} : H\mapsto K^G$ defined by $\hat{\psi}(y)=f_y$, where $f_y(x)=\psi(x,y)$ for all $x\in V(G)$, is also a homomorphism. Conversely, a homomorphism from $H$ to $K^G$ also defines a homomorphism from $G\times H$ to $K$, as indicated in the proof of Theorem 2. It follows that Hedetniemi's conjecture is equivalent to the statement that for every undirected graph

$G$ such that $\chi(G) > n$, we have $\chi(K_n^G) = n$. For $n = 3$, this statement is proved in [2]. However, no general bound for $\chi(K_n^G)$ that uses only the high chromaticity of $G$ is known. In particular, proving that there exists a number $M$ such that for every directed graph $G$, $\chi(G) \geq M$ implies $\chi(K_3^G) \leq M$ is equivalent to proving that the Poljak–Rödl function of the introduction is unbounded. We will see in Section 4 that the fractional version of this problem is much more tractable.

We will use three equivalent definitions of the fractional chromatic number; see [7] for a detailed treatment. Let $\mathcal{I}(G)$ denote the family of all independent sets of a directed graph $G$. A function $\mu : \mathcal{I}(G) \mapsto [0,1]$ is called a *fractional colouring* of $G$ if $\sum_{x \in I} \mu(I) \geq 1$ for all $x \in V(G)$. The value $\sum_{I \in \mathcal{I}(G)} \mu(I)$ is called the *weight* of $\mu$. Also, a function $\nu : V(G) \mapsto [0,1]$ is called a *fractional clique* of $G$ if $\sum_{x \in I} \nu(x) \leq 1$ for all $I \in \mathcal{I}(G)$. Its *weight* is $\sum_{x \in V(G)} \nu(x)$. The *fractional chromatic number* $\chi_f(G)$ of $G$ is the common value of the minimum weight of a fractional colouring of $G$ and the maximum weight of a fractional clique of $G$. In terms of homomorphisms, the fractional chromatic number can also be defined as follows:

$$\chi_f(G) = \min\{\tfrac{s}{r} : G \text{ admits a homomorphism to } K(r,s)\},$$

where $K(r,s)$ is the *Kneser graph* whose vertices are the $r$-subsets of $\{1,\dots,s\}$, where $A \to B$ if and only if $A \cap B = \emptyset$.

Let $G, H$ be graphs such that $\chi_f(G \times H) = \rho$ and $\chi_f(G) \geq 4\rho$. To prove Theorem 2 it suffices to show that $\chi_f(H) \leq 4\rho$. By definition, $\chi_f(G \times H) = \rho$ implies that there exist integers $r, s$ such that $\frac{s}{r} = \rho$ and $G \times H$ admits a homomorphism to $K(r,s)$. Thus $H$ admits a homomorphism to $K(r,s)^G$ and $\chi_f(H) \leq \chi_f(K(r,s)^G)$. We will show that $\chi_f(K(r,s)^G) \leq 4\rho$.

For $x \in V(G)$ and $1 \leq k \leq s$, put

$$I(x,k) = \{h \in K(r,s)^G : k \in h(x) \cap h(y) \text{ for some } x \to y \text{ in } G\}.$$

If $h \in I(x,k)$ and $h' \to h$ in $K(r,s)^G$, then for some $x \to y$ in $G$ we have $k \in h(y)$, thus $k \notin h'(x)$, and $h' \notin I(x,k)$. This shows that $I(x,k)$ is an independent set.

Let $\nu : V(G) \mapsto [0,1]$ be a fractional clique of weight $\chi_f(G)$. For $x \in V(G)$ and $1 \leq k \leq s$, put

$$\mu(I(x,k)) = \tfrac{4}{r \cdot \chi_f(G)} \nu(x)$$

(and $\mu(I) = 0$ for all other $I \in \mathcal{I}(K(r,s)^G)$). Then $\sum_{I \in \mathcal{I}(K(r,s)^G)} \mu(I) = 4\rho$. We will show that $\mu$ is a fractional colouring of $K(r,s)^G$.

For a function $h \in V(K(r,s)^G)$, let $G_h$ be the subgraph of $G$ induced by

$$V(G_h) = \{x \in V(G) : |h(x) \cap (\cup_{x \to y} h(y))| \leq \tfrac{r}{2}\}.$$

For every $x \in V(G_h)$, we can select a set $A(x) \subseteq h(x)$ such that $|A(x)| = \lceil \frac{r}{2} \rceil$ and $A(x) \cap h(y) = \emptyset$ for all $x \to y$ in $G$. The map $\psi : G_h \mapsto K(\lceil \frac{r}{2} \rceil, s)$ defined by $\psi(x) = A(x)$ is a homomorphism, therefore $\chi_f(G_h) \leq \frac{2s}{r} \leq \frac{\chi_f(G)}{2}$. Since the restriction of $\nu$ to $V(G_h)$ is a fractional clique of $G_h$, we then have $\sum_{x \in V(G_h)} \nu(x) \leq \frac{\chi_f(G)}{2}$. Therefore

$$\begin{aligned}
\sum_{h \in I} \mu(I) &= \sum_{x \in V(G)} \sum \Big\{ \mu(I(x,k)) : k \in h(x) \cap h(y) \text{ for some } x \to y \Big\} \\
&\geq \sum_{x \notin V(G_h)} \sum \Big\{ \tfrac{4}{r \cdot \chi_f(G)} \cdot \nu(x) : k \in h(x) \cap h(y) \text{ for some } x \to y \Big\} \\
&\geq \sum_{x \notin V(G_h)} \tfrac{4}{r \cdot \chi_f(G)} \cdot \nu(x) \cdot \tfrac{r}{2} \\
&\geq \tfrac{4}{r \cdot \chi_f(G)} \cdot \tfrac{\chi_f(G)}{2} \cdot \tfrac{r}{2} = 1.
\end{aligned}$$

This shows that $\mu$ is a fractional colouring of $K(r,s)^G$ of weight $4\rho$, and concludes the proof of Theorem 1. ∎

## 4. The fractional Poljak–Rödl function

In this section, we consider the fractional analogue of the Poljak–Rödl function presented in the introduction:

$$\phi_f(x) = \inf\{\chi_f(G \times H) : G, H \text{ are directed graphs and } \chi_f(G) \geq \chi_f(H) \geq x\}.$$

The best possible bound for the fractional chromatic number of a categorical product of directed graphs in terms of the fractional chromatic numbers of its factors is therefore

$$\chi_f(G \times H) \geq \phi_f\left(\min\{\chi_f(G), \chi_f(H)\}\right).$$

We will show that this bound is essentially linear:

**Theorem 3.** $\frac{\phi_f(x)}{x}$ *converges to* $c = \inf\{\frac{\phi_f(x)}{x} : x \geq 2\}$.

To prove this result, we introduce the *lexicographic product* $G[H]$ of two directed graphs $G$ and $H$: The vertex set of $G[H]$ is $V(G) \times V(H)$, and $(u,v) \to (u',v')$ in $G[H]$ if and only if $u = u'$ and $v \to v'$, or $u \to u'$. There is a very simple formula expressing the fractional chromatic number of $G[H]$ in terms of those of $G$ and $H$:

**Lemma 4** ([3])**.** $\chi_f(G[H]) = \chi_f(G) \cdot \chi_f(H)$.

(The result is stated only for undirected graphs in [3], but it is valid for directed graphs as well, since the orientation does not affect the structure of the independent sets.) In particular, for the transitive tournament $T_n$ we have $\chi_f(G[T_n]) = n \cdot \chi_f(G)$.

**Lemma 5.** $\chi_f(G[T_n] \times H[T_n]) = n \cdot \chi_f(G \times H)$.

**Proof.** Let $\nu : G \times H \mapsto [0,1]$ be a fractional clique of weight $\chi_f(G \times H)$. We define the function $\nu' : G[T_n] \times H[T_n] \mapsto [0,1]$ by

$$\nu'((u,x),(v,y)) = \begin{cases} \nu(u,v) \text{ if } x = y, \\ 0 \text{ otherwise.} \end{cases}$$

Let $I$ be an independent set of $G[T_n] \times H[T_n]$. Then for every $(u,v) \in V(G \times H)$, there is at most one $x \in V(T_n)$ such that $((u,x),(v,x)) \in I$. Furthermore, the set

$$J = \{(u,v) \in V(G \times H) : \text{there exists } x \in V(T_n) \text{ such that } ((u,x),(v,x)) \in I\}$$

is independent, since $(u',v') \to (u,v)$ in $J$ would imply $((u',x'),(v',x')) \to ((u,x),(v,x))$ in $I$ for some $x', x \in V(T_n)$. Therefore

$$\sum_{((u,x),(v,y)) \in I} \nu'((u,x),(v,y)) \leq \sum_{(u,v) \in J} \nu(u,v) \leq 1.$$

This shows that $\nu'$ is a fractional clique of $G[T_n] \times H[T_n]$, whence $\chi_f(G[T_n] \times H[T_n]) \geq n \cdot \chi_f(G \times H)$.

To prove the converse inequality, we use the following

**Claim.** Let $I$ be an independent set of $G \times H$ and $S_I$ the subgraph of $G[T_n] \times H[T_n]$ induced by

$$V(S_I) = \{((u,x),(v,y)) \in V(G[T_n] \times H[T_n]) : (u,v) \in I\}.$$

Then $S_I$ is $n$-colourable.

**Proof of Claim.** Put

$$J = \{(u,v) \in I : \text{there exists } (u',v) \in I \text{ such that } u' \to u\},$$
$$J' = \{(u,v) \in I : \text{there exists } (u,v') \in I \text{ such that } v' \to v\}.$$

We split $I$ into the four sets $I_{\max} = J \cap J'$, $I_1 = J' \setminus J$, $I_2 = J \setminus J'$ and $I_{\min} = I \setminus (J \cup J')$. We now define $c : S_I \mapsto \{1, \ldots, n\}$ by

$$c((u,x),(v,y)) = \begin{cases} \max\{x,y\} & \text{if } (u,v) \in I_{\max}, \\ x & \text{if } (u,v) \in I_1, \\ y & \text{if } (u,v) \in I_2, \\ \min\{x,y\} & \text{if } (u,v) \in I_{\min}. \end{cases}$$

We show that $c$ is an $n$-colouring of $S_I$. For $((u',x'),(v',y')) \to ((u,x),(v,y))$ in $S_I$, we have $(u',x') \to (u,x)$ in $G[T_n]$ and $(v',y') \to (v,y)$ in $H[T_n]$. Therefore $u'=u$ or $u' \to u$ in $G$ and $v'=v$ or $v' \to v$ in $H$. If $u' \to u$, then since $I$ is independent, we cannot have $v' \to v$, hence $v'=v$ and $y'<y$. This implies that $(u,v) \in J$; moreover there cannot exist a vertex $v''$ in $H$ such that $v'' \to v' = v$ and $(u',v'') \in I$ (for otherwise we would have $(u',v'') \to (u,v)$ in $I$) hence $(u',v') \notin J'$. We then have

$$c((u',x'),(v',y')) \le y' < y \le c((u,x),(v,y)).$$

Similarly, $v' \to v$ implies

$$c((u',x'),(v',y')) \le x' < x \le c((u,x),(v,y)).$$

The remaining possibility is $u'=u, v'=v$. We then have $x'<x$ and $y'<y$. Since the functions max, min and the projections are all proper $n$-colourings of $T_n \times T_n$, we again have $c((u',x'),(v',y')) \ne c((u,x),(v,y))$. Therefore $S_I$ is $n$-colourable, which proves our Claim. ∎

Let $\mu : \mathcal{I}(G \times H) \mapsto [0,1]$ be a fractional colouring of $G \times H$. By the previous Claim, for any $I \in \mathcal{I}(G \times H)$, the set $S_I$ can be partitioned into $n$ sets $S_I(1), \dots, S_I(n) \in \mathcal{I}(G[T_n] \times H[T_n])$. Thus we can define a fractional colouring $\mu' : \mathcal{I}(G[T_n] \times H[T_n]) \mapsto [0,1]$ by putting $\mu'(S_I(i)) = \mu(I), i = 1, \dots, n$. The weight of $\mu'$ is $n$ times that of $\mu$, therefore $\chi_f(G[T_n] \times H[T_n]) \le n \cdot \chi_f(G \times H)$. ∎

**Proof of Theorem 3.** The function $\phi_f$ is clearly nondecreasing, and by Lemmas 4, 5, we have $\phi_f(n \cdot x) \le n \cdot \phi_f(x)$. It is well known that these conditions imply that $\frac{\phi_f(x)}{x}$ converges to $\inf\{\frac{\phi_f(x)}{x} : x \ge 2\}$. ∎

The precise value of $c$ remains to be determined; for the moment, only the bounds $\frac{1}{4} \le c \le \frac{2}{3}$ are known. Note that the fractional Poljak–Rödl function $\phi_f$ admits an undirected analog, which can also be shown to be essentially linear. However, no improvement on the bound of Theorem 1 is known for undirected graphs. It would be interesting to find a way to use the symmetry of the edges to improve this bound. The insight gained could even help to solve the non-fractional version of the problem.

**Added in Proof.** We learned that Theorem 2 has also been obtained by S. Bessy and S. Thomassé.

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